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On sum and stability of g-frames in Hilbert spaces

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ABSTRACT
In this paper, we give some new results on sum and stability of g-frames in Hilbert spaces. Since the finite sum of g-frames may not be a g-frame for the Hilbert space, we give a necessary and sufficient condition and some sufficient conditions for the finite sum of g-frames to be a g-frame. We also show that every g-sequence in Hilbert space can be expanded to a tight g-frame by adding a linear bounded operator. Moreover, we obtain some sufficient conditions under which g-frames (and the finite sum of g-frames) are stable under small perturbations.

1. Introduction

Frames in Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [1] as part of their research in non-harmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [2], and popularized from then on. Today, frame has been a useful tool in many areas such as characterizing function spaces [2], signal and image processing [3], wireless communications [4,5], probability statistics [6,7] and coding theory [8].

Sun in [9] introduced the concept of g-frame in Hilbert space. G-frames are natural generalizations of frames which cover many other recent generalizations of frames such as bounded quasi-projections [10], fusion frames [11] and pseudo-frames [12].

There is a long tradition for studying the stability of various notions under perturbation. In [13], the author studied the stability of g-frames and dual g-frames. Then the authors in [14] gave the definition of perturbation of g-frames which is based on the perturbation results on frames [15]. The stability of frames is also developed by many other authors, see [16] for frames and [17–19] for g-frames.

In this paper, we give some new characterizations of stability of g-frames from the view of the finite sum of g-frames. We also give some new results about the finite sum of g-frames. Our results are different from the results in [20,21]. Moreover, we prove that every g-sequence can be extended to a tight g-frame from the view of frame theory.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i\}_{i \in I}$ is a sequence of closed subspaces of $\mathcal{K}$, where $I$ is a subset of $\mathbb{Z}$ and $L(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_i$. For $T \in L(\mathcal{H})$, we denote $T^\dagger$ for pseudo-inverse of $T$. And we denote by $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$. A sequence $\{a_i\}_{i \in I}$ is said to be positively confined if $0 < \inf_{i \in I} a_i \leq \sup_{i \in I} a_i < +\infty$. 

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Note that for any sequence \( \{H_i\}_{i \in I} \), we can assume that there exists a Hilbert space \( K \) such that for all \( i \in I \), \( H_i \subset K \) (for example \( K = \bigoplus_{i \in I} H_i \)).

**Definition 1.1:** A sequence \( \{\Lambda_i\} \subset L(H, H_i) \) of bounded operators from \( H \) to \( H_i \) is said to be a generalized frame, or simply a g-frame, for \( H \) with respect to \( \{H_i\}_{i \in I} \) if there are two positive constants \( A \) and \( B \) such that

\[
A \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B \|f\|^2, \quad \forall f \in H.
\]

We call \( A \) and \( B \) the lower and upper g-frame bounds, respectively. We call \( \{\Lambda_i\}_{i \in I} \) a tight g-frame if \( A = B \) and a Parseval g-frame if \( A = B = 1 \). If only the second inequality is required, we call it a Bessel g-sequence.

For each sequence \( \{H_i\}_{i \in I} \), we define the space \( \left( \sum_{i \in I} \oplus H_i \right)_{\ell^2} \) by

\[
\left( \sum_{i \in I} \oplus H_i \right)_{\ell^2} = \left\{ \{f_i\}_{i \in I} \mid f_i \in H_i, \sum_{i \in I} \|f_i\|^2 < \infty \right\}.
\]

with the inner product defined by

\[
\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle.
\]

The synthesis operator of \( \{\Lambda_i\}_{i \in I} \) is given by

\[
T_\Lambda : \oplus H_i \longrightarrow H; \quad T_\Lambda \{f_i\}_{i \in I} = \sum_{i \in I} \Lambda_i^* f_i.
\]

We call the adjoint \( T_\Lambda^* \) of the synthesis operator the analysis operator which is given by

\[
T_\Lambda^* f = \{\Lambda_i f\}_{i \in I}.
\]

By composing \( T_\Lambda \) and \( T_\Lambda^* \), we obtain the g-frame operator

\[
S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i \in I} \Lambda_i^* \Lambda_i f
\]

which is bounded, positive and invertible. Then, the following reconstruction formula takes place for all \( f \in H \)

\[
f = S_\Lambda^{-1} S_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f = \sum_{i \in I} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f.
\]

We call \( \{\Lambda_i S_\Lambda^{-1}\}_{i \in I} \) the canonical dual g-frame of \( \{\Lambda_i\}_{i \in I} \).

In [9], the author had shown that every g-frame can be considered as a frame. More precisely, let \( \{\Lambda_i\}_{i \in I} \) be a g-frame for \( H \) and \( \{e_{ij}\}_{j \in J_i} \) be an orthonormal basis for \( H_i \), then there exists a frame \( \{u_{ij}\}_{i \in I, j \in J_i} \) of \( H \) such that

\[
u_{ij} = \Lambda_i^* e_{ij},
\]
Lemma 1.3 [22]: Then there exists an operator $T$ and $t$ such that

$$\Lambda_i f = \sum_{j \in I_i} \langle f, u_{ij} \rangle e_{ij}, \quad \forall f \in \mathcal{H},$$

and

$$\Lambda_i^* g = \sum_{j \in I_i} \langle g, e_{ij} \rangle u_{ij}, \quad \forall g \in \mathcal{H}_i.$$  

We call $\{u_{ij}\}_{i \in I, j \in I_i}$ the frame induced by $\{\Lambda_i\}_{i \in I}$ with respect to $\{e_{ij}\}_{i \in I, j \in I_i}$. The next lemma is a characterization of g-frame by a frame.

Lemma 1.2 [9]: Let $\{\Lambda_i\}_{i \in I}$ be a family of linear operators and $u_{ij}$ be defined as in (1). Then $\{\Lambda_i\}_{i \in I}$ is a g-frame (tight g-frame) for $\mathcal{H}$ if and only if $\{u_{ij}\}_{i \in I, j \in I_i}$ is a frame (tight frame) for $\mathcal{H}$.

The following lemmas are key tools for the proofs of our main results.

Lemma 1.3 [22]: Let $\mathcal{H}$ be a Hilbert space, and suppose that $T \in L(\mathcal{H})$ has a closed range. Then there exists an operator $T^+ \in L(\mathcal{H})$ for which

$$\mathcal{N}(T^+) = \mathcal{R}(T)\perp, \quad \mathcal{R}(T^+) = \mathcal{N}(T)\perp, \quad TT^+ f = f, \quad f \in \mathcal{R}(T).$$

We call the operator $T^+$ the pseudo-inverse of $T$. If $T$ is invertible, then we have $T^{-1} = T^+$.

Lemma 1.4 [23]: (Cauchy inequality) Let $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_m)$ be two sequences of real numbers, then

$$\left(\sum_{i=1}^{m} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{m} a_i^2 \right) \left(\sum_{i=1}^{m} b_i^2 \right),$$

with equality if and only if the sequence $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_m)$ are proportional.

2. The sums of g-frames

Let $\{\Lambda_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$ be two g-frames for $\mathcal{H}$, then $\{\Lambda_i + \Gamma_i\}_{i \in I}$ may not be a g-frame for $\mathcal{H}$. In fact, if we take $\Gamma_i = -\Lambda_i$, it is easy to find that $\{\Lambda_i + \Gamma_i\}_{i \in I}$ is not a g-frame. If we take $\Lambda_i = \Theta_i$ for all $i \in I$ and $\Gamma_i = 0, \Gamma_i = \Theta_i (i \neq j)$, where $\{\Theta_i\}_{i \in I}$ is a g-orthonormal basis for $\mathcal{H}$. Then $\{\Lambda_i + \Gamma_i\}_{i \in I}$ is a g-frame for $\mathcal{H}$. However, $\{\Gamma_i\}_{i \in I}$ is not a g-frame but a g-sequence for $\mathcal{H}$.

It is natural to ask for condition under which $\{\Lambda_i + \Gamma_i\}_{i \in I}$ is a g-frame for $\mathcal{H}$. We first give a characterization of sum of g-frames with positively confined sequences.

Theorem 2.1: Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for $\mathcal{H}$ with bounds $A, B$. Let $\{\Gamma_i\}_{i \in I}$ be a g-sequence with synthesis operator $T\Gamma$. For any two positively confined sequences $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I}$, if

$$\|T\Gamma\|^2 \leq \frac{A^{\inf}_{i \in I} a_i^2}{2 \sup_{i \in I} b_i^2},$$

then $\{a_i \Lambda_i + b_i \Gamma_i\}_{i \in I}$ is a g-frame for $\mathcal{H}$.
Proof: For any $f \in \mathcal{H}$, we have
\[
\sum_{i \in I} \|a_i \Lambda_i + b_i \Gamma_i \|^2 = \sum_{i \in I} \|a_i \Lambda_i f\|^2 + \sum_{i \in I} \|b_i \Gamma_i f\|^2 + 2 \text{Re} \sum_{i \in I} (a_i \Lambda_i f, b_i \Gamma_i f)
\]
\[
\leq 2 \left( \sum_{i \in I} \|a_i \Lambda_i f\|^2 + \sum_{i \in I} \|b_i \Gamma_i f\|^2 \right)
\]
\[
\leq 2 \left( \left( \sup_{i \in I} a_i^2 \right) \sum_{i \in I} \|\Lambda_i f\|^2 + \left( \sup_{i \in I} b_i^2 \right) \sum_{i \in I} \|\Gamma_i f\|^2 \right)
\]
\[
\leq 2 \left( \left( \sup_{i \in I} a_i^2 \right) B \|f\|^2 + \left( \sup_{i \in I} b_i^2 \right) \|T_{\Gamma_i} f\|^2 \right)
\]
\[
\leq 2 \left( \sup_{i \in I} a_i^2 \right) B + \left( \sup_{i \in I} b_i^2 \right) \|T_{\Gamma_i} f\|^2 \|f\|^2.
\]
Since
\[
\sum_{i \in I} \|a_i \Lambda_i f\|^2 = \sum_{i \in I} \|a_i \Lambda_i f - b_i \Gamma_i f\|^2
\]
\[
\leq 2 \left( \sum_{i \in I} \|a_i \Lambda_i f\|^2 + \sum_{i \in I} \|b_i \Gamma_i f\|^2 \right),
\]
we have
\[
2 \sum_{i \in I} \|a_i \Lambda_i + b_i \Gamma_i \|^2 \geq \sum_{i \in I} \|a_i \Lambda_i f\|^2 - 2 \sum_{i \in I} \|b_i \Gamma_i f\|^2
\]
\[
\geq \left( \inf_{i \in I} a_i^2 \right) \sum_{i \in I} \|\Lambda_i f\|^2 - 2 \left( \sup_{i \in I} b_i^2 \right) \|T_{\Gamma_i} f\|^2
\]
\[
\geq \left( \inf_{i \in I} a_i^2 \right) A - 2 \left( \sup_{i \in I} b_i^2 \right) \|T_{\Gamma_i} f\|^2 \|f\|^2 > 0.
\]
Hence, \( \{a_i \Lambda_i + b_i \Gamma_i\}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

Let \( a_i = b_i = 1 \), we have the following corollary.

Corollary 2.2: Let \( \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with bounds \( A, B \). Let \( \{\Gamma_i\}_{i \in I} \) be a g-sequence with synthesis operator \( T_{\Gamma} \). If \( \|T_{\Gamma} f\|^2 < \frac{A}{2} \), then \( \{\Lambda_i + \Gamma_i\}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

When \( \{\Gamma_i\}_{i \in I} \) is Bessel g-sequence, we give a sufficient condition under which \( \{\Lambda_i + \Gamma_i\}_{i \in I} \) is a g-frame.

Corollary 2.3: Let \( \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with bounds \( A, B \) and frame operator \( S_{\Lambda} \). Let \( \{\Gamma_i\}_{i \in I} \) be a Bessel g-sequence for \( \mathcal{H} \) with Bessel bound \( M \). For any non-zero constant \( a, b \), if \( |b|^2 < \frac{|a|^2 A^2}{2M} \|S_{\Lambda}\|^{-1} \), then \( \{a \Lambda_i + b \Gamma_i\}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

Proof: This follows immediately from Theorem 2.1 when we take \( |a| = \inf a_i \) and \( |b| = \sup b_i \) for all \( i \in I \).

Next, we consider \( \{\Lambda_i + \Gamma_i\}_{i \in I} \) with bounded operators when \( \{\Lambda_i\}_{i \in I} \) and \( \{\Gamma_i\}_{i \in I} \) are two g-frames for \( \mathcal{H} \).
**Theorem 2.4:** Let \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) be two g-frames for \( \mathcal{H} \), and let \( T_\Lambda \) and \( T_\Gamma \) be synthesis operators of \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \), respectively. Let \( U, V \in L(\mathcal{H}) \). If \( T_\Lambda T_\Gamma^* = 0 \) and \( U \) or \( V \) is surjective, then \( \{ \Lambda_i U + \Gamma_i V \}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

**Proof:** Since \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) are two g-frames, there exist \( 0 < A_1 \leq B_1 < \infty \) and \( 0 < A_2 \leq B_2 < \infty \) such that

\[
A_1 \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B_1 \|f\|^2, \quad A_2 \|f\|^2 \leq \sum_{i \in I} \|\Gamma_i f\|^2 \leq B_2 \|f\|^2.
\]

Since \( T_1 T_2^* = 0 \), for any \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I} \Lambda_i^* \Gamma_i f = \sum_{i \in I} \Gamma_i^* \Lambda_i f = 0.
\]

Hence, for all \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I} \|(\Lambda_i U + \Gamma_i V)f\|^2 = \sum_{i \in I} \langle \Lambda_i U + \Gamma_i Vf, \Lambda_i U + \Gamma_i Vf \rangle \\
= \sum_{i \in I} \|\Lambda_i Uf\|^2 + \sum_{i \in I} \|\Gamma_i Vf\|^2 + 2\text{Re} \sum_{i \in I} \langle \Lambda_i^* \Gamma_i Vf, Uf \rangle \\
= \sum_{i \in I} \|\Lambda_i Uf\|^2 + \sum_{i \in I} \|\Gamma_i Vf\|^2 \\
\leq B_1 \|Uf\|^2 + B_2 \|Vf\|^2 \leq (B_1 \|U\|^2 + B_2 \|V\|^2) \|f\|^2.
\]

Without loss of generality, assume that \( U \) is surjective. Then there exists some constant \( C > 0 \) such that \( \|Uf\|^2 \geq C\|f\|^2 \) for any \( f \in \mathcal{H} \). Then we have

\[
\sum_{i \in I} \|(\Lambda_i U + \Gamma_i V)f\|^2 = \sum_{i \in I} \|\Lambda_i Uf\|^2 + \sum_{i \in I} \|\Gamma_i Vf\|^2 \\
\geq \sum_{i \in I} \|\Lambda_i Uf\|^2 \geq A_1 \|Uf\|^2 \\
\geq A_1 C\|f\|^2.
\]

So \( \{ \Lambda_i U + \Gamma_i V \}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

\( \square \)

When \( U = 0 \) and \( V \) is surjective, we have the following result.

**Corollary 2.5:** Let \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) be two g-frames for \( \mathcal{H} \), and let \( T_\Lambda \) and \( T_\Gamma \) be synthesis operators of \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \), respectively. Let \( V \in L(\mathcal{H}) \). If \( T_\Lambda T_\Gamma^* = 0 \) and \( V \) is surjective, then \( \{ \Lambda_i + \Gamma_i V \}_{i \in I} \) is a g-frame for \( \mathcal{H} \). Moreover, \( \{ \Lambda_i + \Gamma_i V^a \}_{i \in I} \) is also a g-frame for \( \mathcal{H} \) for any natural numbers \( a \).

The following corollary can be immediately from Theorem 2.4 when \( U = V = I_\mathcal{H} \).

**Corollary 2.6:** Let \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) be two g-frames for \( \mathcal{H} \), and let \( T_\Lambda \) and \( T_\Gamma \) be synthesis operators of \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \), respectively. If \( T_\Lambda T_\Gamma^* = 0 \), then \( \{ \Lambda_i + \Gamma_i V \}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

The next theorem provides a necessary and sufficient condition for that the new g-frame is a tight g-frame.
\textbf{Theorem 2.7:} Let \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) be two Parseval g-frames for \( \mathcal{H} \), and let \( T_\Lambda \) and \( T_\Gamma \) be synthesis operators of \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \), respectively, such that \( T_\Lambda T_\Gamma^* = 0 \). Let \( U, V \in L(\mathcal{H}) \). \( \{ \Lambda_i + \Gamma_i \}_{i \in I} \) is a \( \lambda \)-tight g-frame for \( \mathcal{H} \) if and only if \( U^* U + V^* V = \lambda I_\mathcal{H} \).

\textbf{Proof:} In fact, since \( T_\Lambda T_\Gamma^* = 0 \), we have

\[
\sum_{i \in I} \| (\Lambda_i U + \Gamma_i V) f \|^2 = \sum_{i \in I} \| \Lambda_i U f \|^2 + \sum_{i \in I} \| \Gamma_i V f \|^2
\]

\[
= \| U f \|^2 + \| V f \|^2 = \left\langle U f, U f \right\rangle + \left\langle V f, V f \right\rangle
\]

It follows that \( \{ \Lambda_i U + \Gamma_i V \}_{i \in I} \) is a \( \lambda \)-tight g-frame for \( \mathcal{H} \) if and only if \( U^* U + V^* V = \lambda I_\mathcal{H} \).

Let \( \{ \Lambda_i \}_{i \in I}, l = 1, 2, \ldots, m \) be \( m \) g-frames for \( \mathcal{H} \). Next, we give a necessary and sufficient condition for the finite sum of g-frames to be a g-frame.

\textbf{Theorem 2.8:} Let \( \{ \Lambda_i \}_{i \in I}, l = 1, 2, \ldots, m \) be a g-frame for \( \mathcal{H} \). Let \( \{ \alpha_i \}, l = 1, 2, \ldots, m \) be any scalars. Then \( \{ \sum_{i=1}^{m} \alpha_i \Lambda_i \}_{i \in I} \) is a g-frame if and only if there exists \( \beta > 0 \) and some \( k \in \{ 1, 2, \ldots, m \} \) such that

\[
\beta \sum_{i \in I} \| \Lambda_{k i} f \|^2 \leq \sum_{i \in I} \left\| \sum_{l=1}^{m} \alpha_l \Lambda_{l i} f \right\|^2, \quad \forall f \in \mathcal{H}.
\]

\textbf{Proof:} Let \( \{ \sum_{i=1}^{m} \alpha_i \Lambda_i \}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with bounds \( A, B \), and for any \( k \in \{ 1, 2, \ldots, m \} \), let \( \{ \Lambda_{k i} \}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with bounds \( A_k \) and \( B_k \). Then we have

\[
A \| f \|^2 \leq \sum_{i \in I} \left\| \sum_{l=1}^{m} \alpha_l \Lambda_{l i} f \right\|^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H},
\]

and

\[
\| f \|^2 \geq \frac{1}{B_k} \sum_{i \in I} \| \Lambda_{k i} f \|^2, \quad \forall f \in \mathcal{H}.
\]

Hence

\[
\sum_{i \in I} \left\| \sum_{l=1}^{m} \alpha_l \Lambda_{l i} f \right\|^2 \geq A \| f \|^2 \geq \frac{A}{B_k} \sum_{i \in I} \| \Lambda_{k i} f \|^2 = \beta \sum_{i \in I} \| \Lambda_{k i} f \|^2, \quad \forall f \in \mathcal{H},
\]

where \( \beta = \frac{A}{B_k} \).

Conversely, Since

\[
\beta \sum_{i \in I} \| \Lambda_{k i} f \|^2 \leq \sum_{i \in I} \left\| \sum_{l=1}^{m} \alpha_l \Lambda_{l i} f \right\|^2, \quad \forall f \in \mathcal{H},
\]

for all \( f \in \mathcal{H} \), we have
Lemma 2.9: There is a tight frame for a n-dimensional Hilbert space. We first see the result on frames. All authors in [24] had shown that every Bessel g-sequence (and therefore every g-frame) can be expanded to a tight g-frame by adding some elements.

Theorem 2.10: Let \( \{\Lambda_i\}_{i \in I} \) be a g-sequence for n-dimensional Hilbert space \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \). Then \( \{\Lambda_i\}_{i \in I} \) can be extended to a tight g-frame with \( \Lambda_0 \in L(\mathcal{H}, \mathcal{H}_0) \) if and only if the following inequality is satisfied:

\[
\max_{1 \leq i \leq m} \{a_i^2\} \leq \frac{1}{n} \sum_{i=1}^{m} a_i^2.
\]

and by Cauchy inequality,

\[
\sum_{i=1}^{m} \left( \sum_{l=1}^{m} |a_l|^2 \right)^2 \leq m \left( \sum_{i=1}^{m} \|\Lambda_l f\|^2 \right)^2 \leq m \left( \max_{1 \leq i \leq m} |a_i|^2 \right) \left( \sum_{i=1}^{m} B_i \right) \|f\|^2 \leq B\|f\|^2,
\]

where \( B = m^2 \max\{|a|^2 B_i\} \). Hence, \( \{\sum_{i=1}^{m} a_i \Lambda_i \}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

Let \( \{f_i\}_{i \in I} \) be an A-tight frame for \( \mathcal{H} \), then by putting \( \mathcal{H}_i = \mathbb{C} \) and \( \Lambda(\cdot) = \{\cdot, f_i\} \) for \( i \in I \setminus j \), and \( \Lambda_j = 0 \). Given the g-sequence \( \{\Gamma_i\}_{i \in I} \) with \( \Gamma(\cdot) = \{\cdot, f_j\} \) and \( \Gamma_i = 0 \) for all \( i \in I \setminus j \), it is easy to see that the family \( \{\Lambda_i + \Gamma_i\}_{i \in I} \) is a A-tight g-frame for \( \mathcal{H} \). The authors in [24] had shown that every Bessel g-sequence (and therefore every g-frame) can be expanded to a tight g-frame by adding some elements.

We end this section by giving the following results concerning the expansions of tight g-frames from the view of frame theory. The g-sequence in our result is not necessary a Bessel g-sequence or g-frame. We first see the result on frames.

Lemma 2.9: There is a tight frame for a n-dimensional Hilbert space \( \mathcal{H} \) with m vectors having norms \( \|x_i\| = a_i \), \( i = 1, 2, \ldots, m \) if and only if the following inequality is satisfied:

\[
\max_{1 \leq i \leq m} \{a_i^2\} \leq \frac{1}{n} \sum_{i=1}^{m} a_i^2.
\]

Theorem 2.10: Let \( \{\Lambda_i\}_{i \in I} \) be a g-sequence for n-dimensional Hilbert space \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \). Then \( \{\Lambda_i\}_{i \in I} \) can be extended to a tight g-frame with \( \Lambda_0 \in L(\mathcal{H}, \mathcal{H}_0) \) if and only if \( 0 \leq |J_0| < n \) and

\[
\max_{i \in I, j \in I} \{(\Lambda_i^* e_j)^2\} \leq \frac{1}{n - |J_0|} \sum_{i \in I, j \in I} (\Lambda_i^* e_j)^2,
\]

where \( \{e_j\}_{j \in I} \) is an orthonormal basis for \( \mathcal{H}_i \) and \( \{e_0\}_{j \in J_0} \) is an orthonormal basis for \( \mathcal{H}_0 \).

Proof: Let \( u_{ij} = \Lambda_i^* e_j \) for all \( i \in I, j \in I_i \) and \( u_{0j} = \Lambda_0^* e_0 \). If \( \{u_{ij}\}_{i \in I, j \in I_i} \) is a tight frame for \( \mathcal{H} \), by Lemma 1.2, we have that \( \{\Lambda_i\}_{i \in I} \) is a tight g-frame for \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \). In this case, we have \( \Lambda_0 = 0 \). Assume that \( \{u_{ij}\}_{i \in I, j \in I_i} \) is not a tight frame, by Lemma 2.9, we have

\[
\max_{i \in I, j \in I_i} \{|u_{ij}|^2\} > \frac{1}{n} \sum_{i \in I, j \in I_i} |u_{ij}|^2.
\]

Now we consider \( \{u_{ij}\}_{i \in I, j \in I_i} \cup \{u_{0j}\}_{j \in J_0} \). Then the left side and right side of (2) become

\[
\phi = \max \left\{ \max_{i \in I, j \in I_i} \{|u_{ij}|^2\}, \max_{j \in J_0} \{|u_{0j}|^2\} \right\}, \quad \varphi = \frac{1}{n} \left( \sum_{i \in I, j \in I_i} |u_{ij}|^2 + \sum_{j \in J_0} |u_{0j}|^2 \right).
\]

As we want \( \phi \) to be as small as possible and \( \varphi \) to be as large as possible. So we may assume \( |u_{0j}| = |u_{0k}| = a \) for all \( j, k \in J_0 \). Let \( \max_{i \in I, j \in I_i} \{|u_{ij}|^2\} = b \) and \( \sum_{i \in I, j \in I_i} |u_{ij}|^2 = c \),

\[
\sum_{i \in I} \sum_{l=1}^{m} |a_l|^2 \|\Lambda_l f\|^2 \geq \beta \sum_{i \in I} \|\Lambda_k f\|^2 \geq \beta A_k \|f\|^2,
\]
then we have \( \phi = \max\{a, b\} \) and \( \varphi = \frac{1}{n}(c + |J_0|a^2) \). Consider the function \( \omega(a) = \varphi - \phi : [0, +\infty) \rightarrow \mathbb{R} \) as
\[
\omega(a) = \begin{cases} 
\frac{1}{n}(c + |J_0|a^2) - b^2 & \text{if } b^2 > a^2 \\
\frac{1}{n}(c + |J_0|a^2) - a^2 & \text{if } b^2 \leq a^2 
\end{cases}.
\]
The sequence \( \{u_{ij}\}_{i \in I, j \in J_i} \cup \{u_{0j}\}_{j \in J_0} \) is a tight frame if and only if there exists \( a_0 \) such that \( \omega(a_0) \geq 0 \) by Lemma 2.9. In other words, we need the global maximum of \( \omega \) is nonnegative. Since \( \omega(a) \) is a piecewise monotone function, the global maximum occurs at either 0, \( b^2 \) or \( \infty \). And we can find that
\[
\omega(0) = \frac{c}{n} - b^2 < 0, \quad \omega(b^2) = \frac{c}{n} + \left(\frac{|J_0|}{n} - 1\right)b^2,
\]
and
\[
\lim_{a \to \infty} \omega(a) = \lim_{a \to \infty} \frac{c + (|J_0| - n)a^2}{n} = \begin{cases} 
-\infty & \text{if } |J_0| < n \\
\frac{c}{n} & \text{if } |J_0| = n \\
\infty & \text{if } |J_0| > n
\end{cases}.
\]
Obviously, \( \omega(a^2) < 0 \) when \( |J_0| < n \) and \( \omega(0) < 0 \). So the \( \varphi - \phi \geq 0 \) if and only if \( \omega(b^2) \geq 0 \), which is equivalent to
\[
\max_{i \in I, j \in J_i} |u_{ij}|^2 = b^2 \leq \frac{c}{n - |J_0|} = \frac{1}{n - |J_0|} \sum_{i \in I, j \in J_i} |u_{ij}|^2.
\]
By Lemma 1.2, we hold the conclusion.

### 3. The stability of g-frames

In this section, we study the stability of g-frames. Recall the Corollary 2.2, Let \( \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \), and let \( \{\Gamma_i\}_{i \in I} \) be a g-sequence such that \( \{\Lambda_i + \Gamma_i\}_{i \in I} \) is a Bessel g-sequence with Bessel bound \( D \), what is the condition under which \( \{\Gamma_i\}_{i \in I} \) is also a g-frame for \( \mathcal{H} \). In this direction, we first prove the following result.

**Theorem 3.1:** Let \( \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with bounds \( A, B \) and frame operator \( S_{\Lambda} \). Let \( \{\Gamma_i\}_{i \in I} \) be a g-sequence such that \( \{\Lambda_i + \Gamma_i\}_{i \in I} \) is a Bessel g-sequence with Bessel bound \( D \). If \( D < \frac{A^2\|S_{\Lambda}\|^{-1}}{2} \), then \( \{\Gamma_i\}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

**Proof:** For any \( f \in \mathcal{H} \), we have
\[
\sum_{i \in I} \|\Gamma_if\|^2 = \sum_{i \in I} \|(\Gamma_i + \Lambda_i)f - \Lambda_if\|^2 \\
\leq 2 \left( \sum_{i \in I} \|(\Gamma_i + \Lambda_i)f\|^2 + \sum_{i \in I} \|\Lambda_if\|^2 \right) \\
\leq 2(D + B)\|f\|^2,
\]
and by $B^{-1}I_H \leq S^{-1}_A \leq A^{-1}I_H$, 
\[
2 \sum_{i \in I} \|\Gamma_i f\|^2 \geq \sum_{i \in I} \|\Lambda_i f\|^2 - 2 \sum_{i \in I} \|\Gamma_i + \Lambda_i f\|^2 \\
\geq A\|f\|^2 - 2D\|f\|^2 \geq (A^2\|S\|^{-1} - 2D)\|f\|^2 > 0.
\]

Hence $\{\Gamma_i\}_{i \in I}$ is a g-frame for $H$. \hfill \Box

Next, we give a necessary and sufficient condition under a Bessel g-sequence is stable.

**Theorem 3.2:** Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for $H$ with bounds $A$ and $B$. Let $\{\Gamma_i\}_{i \in I}$ be a g-sequence for $H$. Then $\{\Gamma_i\}_{i \in I}$ is a Bessel g-sequence for $H$ if and only if there exists a $\lambda$ such that
\[
\sum_{i \in I} \|\Lambda_i - \Gamma_i f\|^2 \leq \lambda \sum_{i \in I} \|\Lambda_i f\|^2, \ \forall f \in H.
\]

Moreover, let $T_\Lambda$ be the synthesis operator of $\{\Lambda_i\}_{i \in I}$. If $\lambda < \frac{\|T_\Lambda\|^{-2}}{2B}$, then $\{\Gamma_i\}_{i \in I}$ is a g-frame for $H$.

**Proof:** For any $f \in H$, we have
\[
\sum_{i \in I} \|\Gamma_i f\|^2 = \sum_{i \in I} \|\Gamma_i - \Lambda_i f + \Lambda_i f\|^2 \\
\leq 2 \left( \sum_{i \in I} \|\Gamma_i - \Lambda_i f\|^2 + \sum_{i \in I} \|\Lambda_i f\|^2 \right) \leq 2(B + \lambda)\|f\|^2.
\]

Conversely, let $D$ be the bound of the Bessel g-sequence $\{\Gamma_i\}_{i \in I}$. Since
\[
\|f\|^2 \leq \frac{1}{A} \sum_{i \in I} \|\Lambda_i f\|^2, \ \forall f \in H.
\]

Then for any $f \in H$, we have
\[
\sum_{i \in I} \|\Gamma_i f\|^2 \leq D\|f\|^2 \leq \frac{D}{A} \sum_{i \in I} \|\Lambda_i f\|^2, \ \forall f \in H.
\]

Therefore,
\[
\sum_{i \in I} \|\Lambda_i - \Gamma_i f\|^2 \leq 2 \left( \sum_{i \in I} \|\Lambda_i f\|^2 + \sum_{i \in I} \|\Gamma_i f\|^2 \right) \\
\leq 2 \left( 1 + \frac{D}{A} \right) \sum_{i \in I} \|\Lambda_i f\|^2 \\
= \lambda \sum_{i \in I} \|\Lambda_i f\|^2,
\]

where $\lambda = 2\left(1 + \frac{D}{A}\right)$. 


Moreover, let $T_A$ be the synthesis operator of $\{\Lambda_i\}_{i \in I}$, since $T_A$ is onto, by Lemma 1.3, there exists an operator $T_A^+$ such that $T_A T_A^+ = I_H$ with $B^{-1} \leq \|T_A^+\| \leq A^{-1}$. Then for all $f \in \mathcal{H}$,

$$\|f\|^2 = \|T_A T_A^+ f\|^2 \leq \|(T_A^+)^* T_A^+ f\|^2 = \|T_A^+\|^2 \sum_{i \in I} \|\Lambda_i f\|^2,$$

thus

$$\sum_{i \in I} \|\Lambda_i f\|^2 \geq \|T_A^+\|^2 \|f\|^2 \geq \frac{\|T_A^+\|^2}{B} \sum_{i \in I} \|\Lambda_i f\|^2.$$

Since

$$\sum_{i \in I} \|\Lambda_i f\|^2 = \sum_{i \in I} \|(\Lambda_i - \Gamma_i) f + \Gamma_i f\|^2 \leq 2 \left( \sum_{i \in I} \|(\Lambda_i - b_i \Gamma_i) f\|^2 + \sum_{i \in I} \|\Gamma_i f\|^2 \right),$$

by hypotheses, we also have

$$\sum_{i \in I} \|\Gamma_i f\|^2 \geq \frac{1}{2} \left( \sum_{i \in I} \|\Lambda_i f\|^2 - 2 \sum_{i \in I} \|(\Gamma_i - \Lambda_i) f\|^2 \right) \geq \frac{1}{2} \left( \frac{\|T_A^+\|^2}{B} - 2\lambda \right) \sum_{i \in I} \|\Lambda_i f\|^2 > 0$$

Hence $\{\Gamma_i\}_{i \in I}$ is a g-frame for $\mathcal{H}$.

In [13], the authors characterized the stability of g-frames under small perturbation as follows.

**Theorem 3.3:** Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for $\mathcal{H}$ with bounds $A$ and $B$. Suppose that $\Gamma_i \in L(\mathcal{H}, \mathcal{H})$ and there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that $\max\{\lambda_1 + \mu/\sqrt{A}, \lambda_2\} < 1$ and

$$\left( \sum_{i \in I} \|(\Lambda_i - \Gamma_i) f\|^2 \right)^{1/2} \leq \lambda_1 \left( \sum_{i \in I} \|\Lambda_i f\|^2 \right)^{1/2} + \lambda_2 \left( \sum_{i \in I} \|\Gamma_i f\|^2 \right)^{1/2} + \mu \|f\|, \quad \forall f \in \mathcal{H}.$$

Then $\{\Gamma_i\}_{i \in I}$ is a g-frame for $\mathcal{H}$.

Moreover, the authors in [19] gave a new perturbation which is more general than Theorem 3.3.

**Theorem 3.4:** Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for $\mathcal{H}$ with bounds $A$ and $B$. Let $\{c_i\}_{i \in I}$ be an arbitrary sequence of positive numbers such that $\sum_{i \in I} c_i^2 < \infty$. Suppose that $\Gamma_i \in L(\mathcal{H}, \mathcal{H})$ and there exist constants $\lambda_1, \lambda_2$ such that $(1 - \lambda_1)\sqrt{A} > \left( \sum_{i \in I} c_i^2 \right)^{1/2}$ and

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda_1 \|\Lambda_i f\| + \lambda_2 \|\Gamma_i f\| + c_i \|f\|, \quad \forall f \in \mathcal{H}.$$

Then $\{\Gamma_i\}_{i \in I}$ is a g-frame for $\mathcal{H}$.

We give a sufficient condition under which a g-frame is stable under small perturbations in terms of positively confined sequence.

**Theorem 3.5:** Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for $\mathcal{H}$ and let $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I} \subset \mathbb{R}$ be two positively confined sequences. Suppose that $\Gamma_i \in L(\mathcal{H}, \mathcal{H})$ and there exist constants $0 \leq \lambda_1, \lambda_2$ such that $\max\{\lambda_1 + \mu/\sqrt{A}, \lambda_2\} < 1$ and

$$\left( \sum_{i \in I} \|(\Lambda_i - \Gamma_i) f\|^2 \right)^{1/2} \leq \lambda_1 \left( \sum_{i \in I} \|\Lambda_i f\|^2 \right)^{1/2} + \lambda_2 \left( \sum_{i \in I} \|\Gamma_i f\|^2 \right)^{1/2} + \mu \|f\|, \quad \forall f \in \mathcal{H}.$$
\(\lambda, \mu < 1/2\) such that

\[
\sum_{i \in I} \| (a_i \Lambda_i - b_i \Gamma_i) f \|^2 \leq \lambda \sum_{i \in I} \| a_i \Lambda_i f \|^2 + \mu \sum_{i \in I} \| b_i \Gamma_i f \|^2, \quad f \in \mathcal{H}.
\]

Then \(\{ \Gamma_i \}_{i \in I}\) is a g-frame for \(\mathcal{H}\) with bounds

\[
\frac{(1 - 2\lambda)(\inf_{i \in I} a_i)^2}{2(1 + \mu)(\sup_{i \in I} b_i)^2} A \quad \text{and} \quad \frac{2(1 + \lambda)(\sup_{i \in I} a_i)^2}{(1 - 2\mu)(\inf_{i \in I} b_i)^2} B.
\]

**Proof:** For any \(f \in \mathcal{H}\),

\[
\sum_{i \in I} \| b_i \Gamma_i f \|^2 = \sum_{i \in I} \| (b_i \Gamma_i - a_i \Lambda_i) f + a_i \Lambda_i f \|^2
\]

\[
\leq 2 \left( \sum_{i \in I} \| (a_i \Lambda_i - b_i \Gamma_i) f \|^2 + \sum_{i \in I} \| a_i \Lambda_i f \|^2 \right)
\]

\[
\leq 2 \left( \lambda \sum_{i \in I} \| a_i \Lambda_i f \|^2 + \mu \sum_{i \in I} \| b_i \Gamma_i f \|^2 + \sum_{i \in I} \| a_i \Lambda_i f \|^2 \right),
\]

then we have

\[
(1 - 2\mu) \left( \inf_{i \in I} b_i \right)^2 \sum_{i \in I} \| \Gamma_i f \|^2 \leq 2(1 + \lambda) \left( \sup_{i \in I} a_i \right)^2 \sum_{i \in I} \| \Lambda_i f \|^2.
\]

Thus

\[
\sum_{i \in I} \| \Gamma_i f \|^2 \leq \frac{2(1 + \lambda)(\sup_{i \in I} a_i)^2}{(1 - 2\mu)(\inf_{i \in I} b_i)^2} \sum_{i \in I} \| \Lambda_i f \|^2 \leq \frac{2(1 + \lambda)(\sup_{i \in I} a_i)^2}{(1 - 2\mu)(\inf_{i \in I} b_i)^2} B \| f \|^2, \quad \forall f \in \mathcal{H}.
\]

On the other hand, since

\[
\sum_{i \in I} \| a_i \Lambda_i f \|^2 = \sum_{i \in I} \| (a_i \Lambda_i - b_i \Gamma_i) f + b_i \Gamma_i f \|^2
\]

\[
\leq 2 \left( \sum_{i \in I} \| (a_i \Lambda_i - b_i \Gamma_i) f \|^2 + \sum_{i \in I} \| b_i \Gamma_i f \|^2 \right)
\]

\[
\leq 2 \left( \lambda \sum_{i \in I} \| a_i \Lambda_i f \|^2 + \mu \sum_{i \in I} \| b_i \Gamma_i f \|^2 + \sum_{i \in I} \| b_i \Gamma_i f \|^2 \right),
\]

we have

\[
(1 - 2\lambda) \left( \inf_{i \in I} a_i \right)^2 \sum_{i \in I} \| \Lambda_i f \|^2 \leq 2(1 + \mu) \left( \sup_{i \in I} b_i \right)^2 \sum_{i \in I} \| \Gamma_i f \|^2.
\]
Thus,
\[ \sum_{i \in I} \| \Gamma_{i} f \|^2 \geq \frac{(1 - 2\lambda)(\inf_{i \in I} a_i)^2}{2(1 + \mu)(\sup_{i \in I} b_i)^2} \sum_{i \in I} \| \Lambda_{i} f \|^2 \geq \frac{(1 - 2\lambda)(\inf_{i \in I} a_i)^2}{2(1 + \mu)(\sup_{i \in I} b_i)^2} A \| f \|^2. \]

Hence \{\Gamma_{i}\}_{i \in I} is a g-frame for \mathcal{H} with bounds
\[ \frac{(1 - 2\lambda)(\inf_{i \in I} a_i)^2}{2(1 + \mu)(\sup_{i \in I} b_i)^2} A \text{ and } \frac{(1 + \lambda)(\sup_{i \in I} b_i)^2}{(1 - 2\mu)(\inf_{i \in I} a_i)^2} B. \]

Next, we consider stability of finite sum of g-frames.

**Theorem 3.6:** For \( l \in J = \{1, 2, \ldots, m\} \), let \{\Lambda_{li}\}_{i \in I} be a g-frame for \mathcal{H}. Let \( \Gamma_{li} \in L(\mathcal{H}, \mathcal{H}) \) and \( U : \oplus \mathcal{H}_i \to \oplus \mathcal{H}_i \) be a bounded linear operator such that \( U(\sum_{i \in I} \Gamma_{li} f) = \{\Lambda_{li} f\} \) for some \( l \in J \). If there exists a constant \( \lambda \geq 0 \) such that
\[ \sum_{i \in I} \| (\Lambda_{li} - \Gamma_{li}) f \|^2 \leq \lambda \sum_{i \in I} \| \Lambda_{li} f \|^2, \quad \forall f \in \mathcal{H}, \ l \in J. \]

Then, \( \{\sum_{i \in I} \Gamma_{li}\}_{i \in I} \) is a frame for \mathcal{H}. Further, let \( A_l \) and \( B_l \) be the bounds of \( \{\Lambda_{li}\}_{i \in I} \) for all \( l \in J, \ i \in I \), then the bounds of \( \{\sum_{i \in I} \Gamma_{li}\}_{i \in I} \) are given by
\[ \| U \|^{-2} \min_{l \in J} A_l \text{ and } 2m^2(1 + \lambda) \max_{l \in J} B_l. \]

**Proof:** Let \( A_l \) and \( B_l \) be the bounds of \( \{\Lambda_{li}\}_{i \in I} \) for each \( l \in J \), by Cauchy inequality, then we have
\[ \sum_{i \in I} \left\| \sum_{j \in J} \Gamma_{ij} f \right\|^2 \leq m \sum_{l \in J} \sum_{i \in I} \| \Gamma_{i} f \|^2 = m \sum_{i \in I} \| (\Lambda_{li} - \Gamma_{li}) f + \Lambda_{li} f \|^2 \]
\[ \leq 2m \sum_{l \in J} \left( \sum_{i \in I} \| (\Lambda_{li} - \Gamma_{li}) f \|^2 + \sum_{i \in I} \| \Lambda_{li} f \|^2 \right) \]
\[ \leq 2m(1 + \lambda) \sum_{l \in J} \sum_{i \in I} \| \Lambda_{li} f \|^2 \leq 2m(1 + \lambda) \sum_{l \in J} B_l \| f \|^2 \]
\[ \leq 2m^2(1 + \lambda) \max_{l \in J} B_l \| f \|^2, \]
and
\[ \sum_{i \in I} \| \Lambda_{li} f \|^2 = \sum_{i \in I} \left\| U \sum_{j \in J} \Gamma_{ij} f \right\|^2 \leq \| U \|^2 \sum_{i \in I} \left\| \sum_{j \in J} \Gamma_{ij} f \right\|^2. \]

Thus,
\[ \sum_{i \in I} \left\| \sum_{j \in J} \Gamma_{ij} f \right\|^2 \geq \frac{1}{\| U \|^2} \sum_{i \in I} \| \Lambda_{li} f \|^2 \geq \frac{A_l}{\| U \|^2} \| f \|^2 \geq \| U \|^{-2} \min_{l \in J} A_l \| f \|^2. \]
The following result gives a sufficient condition for stability of finite sum of Bessel g-sequence.

**Theorem 3.7:** For \( l \in J = \{1, 2, \ldots, m\} \). Let \( \{\Lambda_{li}\}_{i \in I} \) be a Bessel g-sequence for \( \mathcal{H} \) with synthesis operator \( T_{\Lambda}^{(l)} \). Let \( \{\Gamma_{li}\}_{i \in I} \) be a g-sequence for \( \mathcal{H} \) such that

\[
\sum_{i \in I} \|(\Lambda_{li} - \Gamma_{li})f\|^2 \leq \lambda \sum_{i \in I} \|\Lambda_{li}f\|^2, \quad \lambda \geq 0, \quad \forall f \in \mathcal{H}, \quad l \in J.
\]

If for some \( k \in J \), there exists \( A_k > 0 \) such that

\[
\sum_{i \in I} \|\Lambda_{ki}f\|^2 \geq A_k \|f\|^2, \quad \forall f \in \mathcal{H}
\]

and

\[
\left(2(m - 1) \sum_{l \neq k} \|T_{\Lambda}^{(l)}\|^2 + 4m\lambda \sum_{l \in J} \|T_{\Lambda}^{(l)}\|^2\right) < A_k.
\]

Then \( \{\sum_{i \in I} \Gamma_{li}\}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

**Proof:** For all \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I} \|\Lambda_{ki}f\|^2 = \sum_{i \in I} \left(\sum_{l \in J} \Lambda_{li} - \sum_{l \neq k} \Lambda_{li}\right) f \right)^2 \leq 2 \sum_{i \in I} \|\Lambda_{li}f\|^2 + 2 \sum_{i \in I} \|\Lambda_{li}f\|^2,
\]

it follows that

\[
\sum_{i \in I} \left(\sum_{l \in J} \Lambda_{li} f \right) \|^2 \geq \frac{1}{2} \left(\sum_{i \in I} \|\Lambda_{ki}f\|^2 - \sum_{i \in I} \|\Lambda_{li}f\|^2\right).
\]

Let \( T_{\Lambda}^{(l)} \) be the synthesis operator of \( \{\Lambda_{li}\}_{i \in I} \), for all \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I} \|\Lambda_{li}f\|^2 = \|(T_{\Lambda}^{(l)})^*f\|^2 \leq \|\Lambda_{li}f\|^2 = \|T_{\Lambda}^{(l)}\|^2 \|f\|^2.
\]

Then by Cauchy inequality, we have

\[
\sum_{i \in I} \left(\sum_{l \in J} \Gamma_{li} f \right) \|^2 \geq \frac{1}{2} \left(\sum_{i \in I} \left(\sum_{l \in J} \Lambda_{li} f \right) \|^2 - 2 \sum_{i \in I} \left(\sum_{l \in J} (\Gamma_{li} - \Lambda_{li})f \right) \|^2 \right)
\]

\[
\geq \frac{1}{2} \left(\frac{1}{2} \sum_{i \in I} \|\Lambda_{ki}f\|^2 - \sum_{i \in I} \|\Lambda_{li}f\|^2 \right)^2 - 2 \sum_{i \in I} \left(\sum_{l \in J} (\Gamma_{li} - \Lambda_{li})f \right) \|^2
\]

\[
\geq \frac{1}{2} \left(\frac{1}{2} \sum_{i \in I} \|\Lambda_{ki}f\|^2 - (m - 1) \sum_{l \neq k \in I} \|\Lambda_{li}f\|^2 - 2m \sum_{l \in J} \sum_{i \in I} \|(\Gamma_{li} - \Lambda_{li})f\|^2\right)
\]

□
\[
\geq \frac{1}{2} \left( \frac{1}{2} A_k \|f\|^2 - (m - 1) \sum_{l \neq k} \| T^{(l)}_\Lambda \|^2 \|f\|^2 - 2m \lambda \sum_{l \in J} \| T^{(l)}_\Lambda \|^2 \|f\|^2 \right) \\
= \frac{1}{4} \left( A_k - 2(m - 1) \sum_{l \neq k} \| T^{(l)}_\Lambda \|^2 - 4m \lambda \sum_{l \in J} \| T^{(l)}_\Lambda \|^2 \right) \|f\|^2 > 0.
\]

On the other hand, let \( B_l \) be the Bessel bound of \( \{ \Lambda_{li} \}_{i \in I} \) for each \( l \in J \). For all \( f \in \mathcal{H} \), we have

\[
\sum_{i \in I} \left\| \sum_{l \in J} \Gamma_{li} f \right\|^2 \leq 2m \sum_{l \in J} \left( \sum_{i \in I} \| (\Lambda_{li} - \Gamma_{li}) f \|^2 + \sum_{i \in I} \| \Lambda_{li} f \|^2 \right) \\
\leq 2m(1 + \lambda) \sum_{l \in J} \sum_{i \in I} \| \Lambda_{li} f \|^2 \leq 2m(1 + \lambda) \sum_{l \in J} B_l \|f\|^2.
\]

Hence \( \{ \sum_{l \in J} \Gamma_{li} \}_{i \in I} \) is a g-frame for \( \mathcal{H} \).

\[ \square \]

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**References**