Geometric similarity invariants of geometric operators

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Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$.

For an open connected subset $\Omega$ of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, Cowen and Douglas introduced the class of operators $B_n(\Omega)$ in their Acta paper [1].

**Definition ($B_n(\Omega)$)**

An operator $T \in B_n(\Omega)$ if for each $w \in \Omega$, is an eigenvalue of the operator $T$ of constant multiplicity $n$, these eigenvectors span the Hilbert space $\mathcal{H}$ and the operator $T - w$, $w \in \Omega$, is surjective.

It was showed that the map $w \to \ker(T - w)$ is holomorphic and $\pi : E_T \to \Omega$, where

$$E_T(w) = \{ \ker(T - w) : w \in \Omega \}, \pi(\ker(T - w)) = w$$

defines a Hermitian holomorphic vector bundle on $\Omega$. 

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First of all, we need introduce some complex geometry notations:
Let $\xi(\Omega)$ be the algebra consist of the $C^\infty$ functions and $\xi^p(\Omega)$ denote the $p$-differential form of $C^\infty$ functions. Thus we have

$$\xi^0(\Omega) = \xi(\Omega), \xi^1(\Omega) = \{f dz + gd\bar{z} : f, g \in \xi(\Omega)\},$$

$$\xi^2(\Omega) = \{f dzd\bar{z}, f \in \xi(\Omega)\}$$

For any vector bundle $E$ which has $C^\infty$ differential structure, let $\xi^p(\Omega, E)$ denotes $p$-differential forms with the coefficients in $E$. Then each element in $\xi^0(\Omega, E)$ is one of sections of $E$. 
The connection $D$ can be regarded as a differential operator which maps $\xi^0(\Omega, E)$ to $\xi^1(\Omega, E)$. Let $\sigma \in E(w)$, and $h = (\langle \sigma_j, \sigma_i \rangle)_{n \times n}$. Then the canonical connection $D$ which keeping the metric and satisfying the following equality:

$$D\left(\sum_{i=1}^{n} f_i \sigma_i\right) = \sum_{i=1}^{n} df_i \otimes \sigma_i + \sum_{i=1}^{n} \sum_{j=1}^{n} f_i \theta_{j,i} \sigma_j$$

where $\theta = h^{-1} \partial h$. And

$$D^2 = d\theta + \theta \wedge \theta = \bar{\partial}(h^{-1} \partial h)$$

then $-\bar{\partial}(h^{-1} \partial h)$ is called as the curvature of $E$ denoted by $K_E$.
Definiton (Second fundamental form)

Let \( T \in B_2(\Omega) \), and \( \sigma_1(w), \sigma_2(w) \in \text{Ker}(T - w) \). Applying the Schmidt orthogonal progress to \( \sigma_1, \sigma_2 \), then we have \( e_1, e_2 \). Suppose

\[
De_1 = D^{1,0}e_1 + D^{0,1}e_2 = \theta_{11}e_1 + \theta_{21}e_2
\]

and \( De_2 = \theta_{12}e_1 + \theta_{22}e_2 \), then \( \theta_{12} = \langle De_2, e_1 \rangle \) is called as the second fundamental form of \( E_T \)
Cowen-Douglas’ Unitary Classification Theorem

For any $T \in B_n(\Omega)$, when $n = 1$, the curvature is the completely unitary invariant. When $n > 1$, then the curvature and it’s covariant partial derivatives are the completely unitary invariants.
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Question 1 (Similarity of Cowen-Douglas Operators)

For the similarity of Cowen-Douglas operators \( A, B \in B_1(\mathbb{D}) \), whether \( A \sim_s B \) if and only if

\[
\lim_{w \to \partial \mathbb{D}} \frac{K_A(w)}{K_B(w)} = 1.
\]

Or can we use some geometric invariants involving curvature to describe the similarity of Cowen-Douglas operators?
D. N. Clark and G. Misra gave a counter example of this conjecture. Let $S_0$ be the backward unilateral shift operator and $T$ be a weighted (backward) shift operator with sequence $\alpha_n = \frac{(\sum_{j=1}^{n} 1/j)^{1/2}}{(\sum_{j=1}^{n+1} 1/j)^{1/2}}$ and
\[
\frac{K_T}{K_{S_0}} = 1 + \left[1 - \ln(1 - |w|^2)\right]^{-1} - |w|^2\left[1 - \ln(1 - |w|^2)\right]^{-2}
\]
Then $\frac{K_T}{K_{S_0}} \rightarrow 1$, when $|w|$ goes to 1. However, $T$ and $S_0$ are not similar.

Let $S$ denote a backward weighted shift operator with weight sequence $\alpha_n = \left[ \frac{(n+1)}{(n+2)} \right] \alpha/2$ and $T$ is a backward weighted shift operator with $\| T \| \leq 1$. Set $\alpha_w$ to be the ratio of the normalized sections of $E_S$ and $E_T$. Then

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(i) $T$ is similar to $S$ if and only if $\alpha_w$ is bounded and bounded from 0.

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(i) $T$ is similar to $S$ if and only if $\alpha_w$ is bounded and bounded from 0.

(ii) $T$ is similar to $S$ with $T = X S X^{-1}$, $X = U + K$ where $U$ is unitary and $K$ is compact if and only if $\alpha_w$ tends to a non-zero limit when $|w| \to 1$. 
In [3], K. Zhu introduced the spanning holomorphic cross-section for Cowen-Douglas operators. Let $T \in B_n(\Omega)$. A holomorphic section of vector bundle $E_T$ is a holomorphic function $\gamma : \Omega \to \mathcal{H}$ such that for each $w \in \Omega$, the vector $\gamma(w)$ belongs to the fibre of $E_T$ over $w$. We say $\gamma$ is a spanning holomorphic section for $E_T$ if $\text{Span} \{ \gamma(w) : w \in \Omega \} = \mathcal{H}$.


For any Cowen-Douglas operator $T \in B_n(\Omega)$, $E_T$ has a spanning holomorphic cross-section. Suppose $T$ and $\tilde{T}$ belongs to $B_n(\Omega)$, then $T$ and $\tilde{T}$ are unitarily equivalent (or similarity equivalent) if and only if there exist spanning holomorphic cross-sections $\gamma_T$ and $\gamma_{\tilde{T}}$ for $E_T$ and $E_{\tilde{T}}$, respectively, such that $\gamma_T \sim_u \gamma_{\tilde{T}}$ (or $\gamma_T \sim_s \gamma_{\tilde{T}}$).

For $T \in B_m(\mathbb{D})$ that is an $n$-hypercontraction, let $P : \mathbb{D} \to \mathcal{L}(\mathcal{H})$ denote the function whose values are orthogonal projections onto $\ker(T - w)$. Then $T$ is similar to $\bigoplus S_n^*$ if and only if there exists a bounded subharmonic function $\psi$ defined on $\mathbb{D}$ such that

$$||\partial P(w)||_2^2 - \frac{mn}{(1 - |w|^2)^2} = \Delta \psi(w),$$
For \( T \in B_m(\mathbb{D}) \) that is an \( n \)-hypercontraction, let \( P : \mathbb{D} \to \mathcal{L}(\mathcal{H}) \) denote the function whose values are orthogonal projections onto \( \ker(T - w) \).

Then \( T \) is similar to \( \bigoplus S_n^* \) if and only if there exists a bounded subharmonic function \( \psi \) defined on \( \mathbb{D} \) such that

\[
\|\partial P(w)\|_2^2 - \frac{mn}{(1 - |w|^2)^2} = \Delta \psi(w),
\]

The Hilbert-Schmidt norm \( \|\partial P(w)\|_2^2 \) is pointed out to be \(-\text{trace} K_T\).
In 1984, G. Misra defined a class of homogeneous Cowen-Douglas operators as the following: an operator $T$ is said to be homogeneous if $\phi(T)$ is unitarily equivalent to $T$ for each Möbius transformation $\phi$, and he proved the following theorem:


Let $T \in B_1(\mathbb{D})$ is a homogenous operator, then $T$ is unitarily equivalent to the adjoint of multiplication operator $M_z$ on the analytic functional space $\mathcal{H}_K$, where $K(z, w) = \frac{1}{(1 - z\bar{w})^\lambda}$, for some $\lambda > -1$.

An operator $T$ is said to be weakly homogeneous if $\phi(T)$ is similarity equivalent to $T$ for each Möbius transformation $\phi$. A natural question is what is the set of all of the weakly homogenous operator at least for Cowen-Douglas class?
Inspired by the structure of homogeneous Cowen-Douglas operators, we introduced the following new class of operators:

**FBₙ(Ω)**

We let $FBₙ(Ω)$ be the set of all bounded linear operators $T$ defined on some complex separable Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$, which are of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix},$$

where the operator $T_i \in B(\mathcal{H}_i)$ is assumed to be in $B_1(Ω)$ and $T_iS_{i,i+1} = S_{i,i+1}T_{i+1}$, $0 \leq i \leq n-2$. 
Second fundamental form in the case of $FB_2(\Omega)$

The $2 \times 2$ block $\begin{pmatrix} T_i & S_{ii+1} \\ 0 & T_{i+1} \end{pmatrix}$ in the decomposition of the operator $T$ in $FB_2(\mathbb{D})$ because of the intertwining property. Hence the corresponding second fundamental form $\theta_{i,i+1}(T)$ is given by the formula

$$\theta_{i,i+1}(T)(z) = \frac{K_{T_i}(z) \, d\bar{z}}{\left( \frac{\|S_{i,i+1}(t_{i+1}(z))\|^2}{\|t_{i+1}(z)\|^2} - K_{T_i}(z) \right)^{1/2}}.$$ (2.1)

**Remark**

For any $T, \tilde{T} \in FB_n(\Omega)$, when $K_{T_i} = K_{\tilde{T}_i}$, then

$$\theta_{i,i+1}(T)(z) = \theta_{i,i+1}(\tilde{T})(z) \iff \frac{\|S_{i,i+1}(t_{i+1}(z))\|}{\|t_{i+1}(z)\|} = \frac{\|\tilde{S}_{i,i+1}(\tilde{t}_{i+1}(z))\|}{\|\tilde{t}_{i+1}(z)\|}$$

So we also use $\frac{\|S_{i,i+1}(t_{i+1}(z))\|}{\|t_{i+1}(z)\|}$ as the second fundamental form $\theta_{i,i+1}(T)$. 
For the unitarily classification problem of Cowen-Douglas operators, we have the following result:

**Theorem 1** [Jiang, Ji, Dinesh and Misra] JFA, 2017

Let $T, \tilde{T} \in \mathcal{FB}_n(\Omega)$.

\[
T \sim_u \tilde{T} \iff \begin{cases} 
K_{T_i} = K_{\tilde{T}_i} \\
\theta_{i,i+1}(T) = \theta_{i,i+1}(\tilde{T}) \\
\frac{\langle S_{i,j}(t_j), t_i \rangle}{\|t_i\|^2} = \frac{\langle \tilde{S}_{i,j}(\tilde{t}_j), \tilde{t}_i \rangle}{\|\tilde{t}_i\|^2}
\end{cases}
\]

Note that numbers of unitarily invariants of common case are $n^2$. But together with the curvature and the second fundamental form, we find a set of $n(n - 1)/2 + 1$ invariants, which are less and easy to compute.
Definition

Let $T \in FB_n(\Omega)$. The operator $T$ is called as quasi-homogeneous operator, i.e. $T \in QB_n(\Omega)$, if $T_i$ is homogenous operator and

$$S_{i,j}(t_j) \in \bigvee \{t^{(k)}_j, k \leq j - i - 1\}.$$  

For the similarity classification of Cowen-Douglas operators, we have the following result:
Similarity of operators in $FB_n(\Omega)$

**Definition**

Let $T \in FB_n(\Omega)$. The operator $T$ is called as quasi-homogeneous operator, i.e. $T \in QB_n(\Omega)$, if $T_i$ is homogenous operator and

$$S_{i,j}(t_j) \in \bigvee \{t_i^{(k)}, k \leq j - i - 1\}.$$

For the similarity classification of Cowen-Douglas operators, we have the following result:

**Theorem 2** [Jiang, Ji and Misra] JFA, 2017

Let $T, S \in QB_n(\Omega)$, then we have

$$\begin{align*}
K_{T_i,i} &= K_{\tilde{T}_i,i} \\
\theta_{i,i+1}(T) &= \theta_{i,i+1}(\tilde{T})
\end{align*}$$

$\implies$ $T \sim_s \tilde{T}$ if and only if $T = \tilde{T}$
We would like to introduce the following a class of geometric operators denoted by $\mathcal{CFB}_n(\Omega)$.

**Definition ($\mathcal{CFB}_n(\Omega)$)**

A geometric operator $T$ with index $n$ is said to be in $\mathcal{CFB}_n(\Omega)$, if $T$ satisfies the following properties:

1. $T$ can be written as an $n \times n$ upper-triangular matrix form $(T_{i,j})_{n \times n}$ under a topological direct decomposition of $H$;
2. $\text{diag}\{T\} = T_{1,1} \circ \cdots \circ T_{n,n} \in \{T\}'$, where $\{T\}'$ denotes the commutant of $T$;
3. each entry $T_{i,j} = \phi_{i,j}T_{i,i+1}T_{i+1,i} \cdots T_{j-1,j}$, where $\phi_{i,j} \in \{T_{i,i}\}'$;
4. $T$ is a strongly irreducible operator, i.e. there are no nontrivial idempotents in $\{T\}'$. 
New progresses

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2. $\text{diag}\{T\} := T_{1,1} + T_{2,2} + \cdots + T_{n,n} \in \{T\}'$, where $\{T\}'$ denotes the commutant of $T$;
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Geometric similarity invariants

Oct 2018
Definition (Similarity invariant set)

Let $\mathcal{F} = \{A_\alpha \in \mathcal{B}(\mathcal{H}), \alpha \in \Lambda\}$. We call $\mathcal{F}$ is a similarity invariant set, if for any invertible operator $X \in \mathcal{B}(\mathcal{H})$,

$$X\mathcal{F}X^{-1} = \{XA_\alpha X^{-1} : A_\alpha \in \mathcal{F}\} = \mathcal{F}.$$
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Proposition

\( CFB_n(\Omega) \) is a similarity invariant set
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\]

Proposition

\( \mathcal{CFB}_n(\Omega) \) is a similarity invariant set

Remark

The set of homogenous operators in Cowen-Douglas class is not a similarity invariant set.
New progresses

**Similarity orbit Theorem (Special case, Apostol, Fialkow, Herrero, and Voiculescu)**

Let $T$ and $S \in B_n(\Omega)$, and spectral pictures of $T$ and $S$ be the same. Then there exist two sequences of invertible operators $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} X_n AX_n^{-1} = B, \quad \lim_{n \to \infty} Y_n BY_n^{-1} = A.$$ 

Notice that $C\mathcal{FB}_n(\Omega)$ is a similarity invariant set, by using the similarity orbit theorem, we can prove that
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\]

Notice that \( \mathcal{CFB}_n(\Omega) \) is a similarity invariant set, by using the similarity orbit theorem, we can prove that

**Theorem**

\( \mathcal{CFB}_n(\Omega) \) is norm dense in \( B_n(\Omega) \).
Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$. Define a Rosenblum operator $\sigma_{T_1, T_2} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ as

$$\sigma_{T_1, T_2}(X) = T_1X - XT_2, \forall X \in \mathcal{L}(\mathcal{H}),$$

and a Rosenblum operator $\sigma_{T_1} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ as

$$\sigma_{T_1}(X) = T_1X - XT_1, \forall X \in \mathcal{L}(\mathcal{H}).$$
New progresses

Definition

Let \( T_1, T_2 \in L(\mathcal{H}) \). Define a Rosenblum operator \( \sigma_{T_1, T_2} : L(\mathcal{H}) \to L(\mathcal{H}) \) as

\[ \sigma_{T_1, T_2}(X) = T_1X - XT_2, \forall X \in L(\mathcal{H}), \]

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\[ \sigma_{T_1}(X) = T_1X - XT_1, \forall X \in L(\mathcal{H}). \]

Definition (Property H)

Let \( T \in CFB_n(\Omega) \). We call \( T \) satisfies the Property (H) if and only if the following statements hold: If \( Y \in B(\mathcal{H}_j, \mathcal{H}_i) \) satisfies

\[ T_{i, i}Y = YT_{i+1, i+1}, \]

\[ Y = T_{i, i}Z - ZT_{i+1, i+1}, \]

for some \( Z, i < j = 1, \ldots, n \).

Then \( Y = 0 \). That is equivalent to \( \ker \sigma_{T_{i, i}, T_{i+1, i+1}} \cap \text{ran} \sigma_{T_{i, i}, T_{i+1, i+1}} = \{0\} \).
Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$. Define a Rosenblum operator $\sigma_{T_1, T_2} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ as

$$\sigma_{T_1, T_2}(X) = T_1X - XT_2, \forall X \in \mathcal{L}(\mathcal{H}),$$

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(ii) $Y = T_{i,i}Z - ZT_{i+1,i+1}$, for some $Z, i < j = 1, \ldots, n.$

Then $Y = 0$. That is equivalent to $\ker \sigma_{T_{i,i}, T_{i+1,i+1}} \cap \text{ran} \sigma_{T_{i,i}, T_{i+1,i+1}} = \{0\}$. 
Proposition

Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and $S_2$ be the right inverse of $T_2$. If $\lim_{n \to \infty} \frac{\|T_1^n\| \cdot \|S_2^n\|}{n} = 0$, then the Property (H) holds i.e. If there exists $X \in \mathcal{L}(\mathcal{H})$ such that $T_1X = XT_2$ and $X = T_1Y - YT_2$ for some $Y$, then $X = 0$. 

Example

Let $A, B \in B_1(\mathcal{D})$ be backward shift operators with weighted sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$. If $\lim_{n \to \infty} n \prod_{k=1}^{n} b_k \prod_{k=1}^{n} a_k = \infty$, then the Property (H) holds.
**Proposition**

Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and $S_2$ be the right inverse of $T_2$. If \( \lim_{{n \to \infty}} \frac{\|T_1^n\| \cdot \|S_2^n\|}{n} = 0 \), then the Property (H) holds i.e. If there exists $X \in \mathcal{L}(\mathcal{H})$ such that $T_1X = XT_2$ and $X = T_1Y - YT_2$ for some $Y$, then $X = 0$.

**Example**

Let $A, B \in B_1(\mathbb{D})$ be backward shift operators with weighted sequences \( \{a_i\}_{i=1}^{\infty} \) and \( \{b_i\}_{i=1}^{\infty} \). If \( \lim_{{n \to \infty}} n \prod_{{k=1}}^{n} \frac{b_k}{a_k} = \infty \), then the Property (H) holds.
New progresses

Definition

We call $T \sim_{U+K} S$, if there exists a unitary operator $U$ and a compact operator $K$ such that $U + K$ is invertible and $(U + K)T = S(U + K)$. 
New progresses

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Lemma

Let $T, S \in B_1(\mathbb{D})$, where $S \sim_u (M_z^*, \mathcal{H}_{K_S}, \mathcal{K}_S)$, then we have

$$T \sim_{U+K} S \iff K_S - K_T = \Delta \ln \phi,$$

where $\phi$ is a bounded function with

$$\phi(w) = 1 + \sum_{i=1}^{m} \frac{2\text{Re}f_i(w)\overline{g}_i(w) + \sum_{i=1}^{m} |g_i(w)|^2}{K_S(w, w)},$$

where $m$ is the rank of $K$ and $\{f_i\}_{i=1}^{m}, \{g_i\}_{i=1}^{m} \in \mathcal{H}_{K_S}$ are orthogonal sets, $\|f_i\| = 1, \|g_i\| \to 0$. When $K_S \geq K_T$, then $\ln \phi$ is subharmonic.
Proposition

Let $A, B \in B_1(\mathbb{D})$ be backward weighted shift operators with weighted sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ respectively. Then the following statements are equivalent:

(i) $A \sim_s B$ implies $A \sim U + K B$,

(ii) $\lim_{n \to \infty} n \prod_{k=1}^{n} a_k \prod_{k=1}^{n} b_k$ exists and is not equal to zero.
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Proposition

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Main Theorem 1 [Jiang and Ji]

Let $T, \tilde{T} \in CFB_n(\Omega)$. Suppose the following statements hold

then we have

$$T \sim_s \tilde{T} \Leftrightarrow \begin{cases} K_{T_i} - K_{\tilde{T}_i} = \Delta \ln \phi_i \\ \frac{\phi_i}{\phi_{i+1}} \theta_{i,i+1}(T) = \theta_{i,i+1}(\tilde{T}) \end{cases}$$

where $\phi_i$ are the bounded subharmonic functions in the Lemma above.
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Let $T, \tilde{T} \in CFB_n(\Omega)$. Suppose the following statements hold

(1) $T$ and $\tilde{T}$ satisfy the Property $(H)$;

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Main Theorem 1 [Jiang and Ji]

Let $T, \tilde{T} \in CFB_n(\Omega)$. Suppose the following statements hold

1. $T$ and $\tilde{T}$ satisfy the Property $(H)$;
2. $T_{i,i} \sim_s \tilde{T}_{i,i}$ implies $T_{i,i} \sim_{U+K} \tilde{T}_{i,i}$

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$$T \sim_s \tilde{T} \iff \begin{cases} K_{T_{i,i}} - K_{\tilde{T}_{i,i}} = \Delta \ln \phi_i \\ \frac{\phi_{i}}{\phi_{i+1}} \theta_{i,i+1}(T) = \theta_{i,i+1}(\tilde{T}) \end{cases}$$

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Definition (Strongly Property (H))

Let $T \in \mathcal{CFB}_n(\Omega)$. We call $T$ satisfies the strongly property (H) if and only if the following statements hold: If $Y \in B(\mathcal{H}_j, \mathcal{H}_i)$ satisfies

Then $Y = 0$. That is equivalent to $\ker \sigma_{T_{i,i}, T_{j,j}} \cap \text{ran} \sigma_{T_{i,i}, T_{j,j}} = \{0\}$. 

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Definition (Strongly Property (H))

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(i) $T_{i,i}Y = YT_{j,j},$

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Strongly Property (H)

Definition (Strongly Property (H))

Let \( T \in \mathcal{CB}_n(\Omega) \). We call \( T \) satisfies the strongly property (H) if and only if the following statements hold: If \( Y \in B(\mathcal{H}_j, \mathcal{H}_i) \) satisfies

(i) \( T_{i,i} Y = YT_{j,j} \),

(ii) \( Y = T_{i,i} Z - ZT_{j,j} \), for some \( Z, i < j = 1, \ldots, n \).

Then \( Y = 0 \). That is equivalent to \( \ker \sigma_{T_{i,i}, T_{j,j}} \cap \text{ran} \sigma_{T_{i,i}, T_{j,j}} = \{0\} \).
Main Theorem 2 [Jiang and Ji]

Let $T = ((T_{i,j}))_{n \times n}$ and $\tilde{T} = ((\tilde{T}_{i,j}))_{n \times n}$ be any two operators in $\mathcal{CFB}_n(\Omega)$, where $T_{i,j} = \tilde{T}_{i,j} = 0, i > j$. Suppose that $T$ satisfies the strongly property (H). Then we have

$$T \sim_s \tilde{T} \iff \begin{cases} \ X_i T_{i,i} = \tilde{T}_{i,i} X_i, \\
X_i T_{i,j} = \tilde{T}_{i,j} X_j, i = 1, 2, \cdots, n 
\end{cases}$$

where $X_i \in \mathcal{L}(\mathcal{H}_i, \tilde{\mathcal{H}}_i), i = 1, 2, \cdots, n$ are invertible operators.
In the following theorem, Soumitra Ghara give a way to decide when an operator $T \in \mathcal{FB}_2(\mathbb{D})$ to be a weakly homogeneous operator.

**Theorem (S. Ghara, Thesis, IISC, 2018)**

Let $1 \leq \lambda \leq \mu \leq \lambda + 2$ and $\psi$ be a non-zero function in $C(\overline{\mathbb{D}}) \cap \text{Hol}(\mathbb{D})$. The operator $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathcal{H}^{(\lambda)} \oplus \mathcal{H}^{(\mu)}$ is weakly homogeneous if and only if $\psi$ is non-vanishing on $\overline{\mathbb{D}}$.

Although the description of the weakly homogeneous operators in $\mathcal{FB}_2(\mathbb{D})$ is more or less clear. However, the computation will become very difficult with the growth of the rank $n$. 
Thus, in general case, we need the intertwining operator between $T$ and $\phi_\alpha(T)$, $\alpha \in \mathbb{D}$ could be diagonal. That means we need to consider the operators in $\mathcal{CFB}_n(\mathbb{D})$ which satisfy the strongly Property $(H)$. In the end of this talk, we will show that there also exists a lot examples of non-weakly homogeneous operators in $\mathcal{CFB}_n(\mathbb{D})$

**Theorem 3 [Jiang and Ji]**

Let $T = \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ 0 & T_{2,2} & T_{2,3} \\ 0 & 0 & T_{3,3} \end{pmatrix} \in \mathcal{CFB}_3(\mathbb{D})$. If $T$ satisfies the strongly Property $(H)$, then $T$ is not weakly homogeneous.
Thank you!!

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